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ON THE SPEED OF PROPAGATION OF WAVES IN A DEFORMED ELASTIC MATE--ETC(U)
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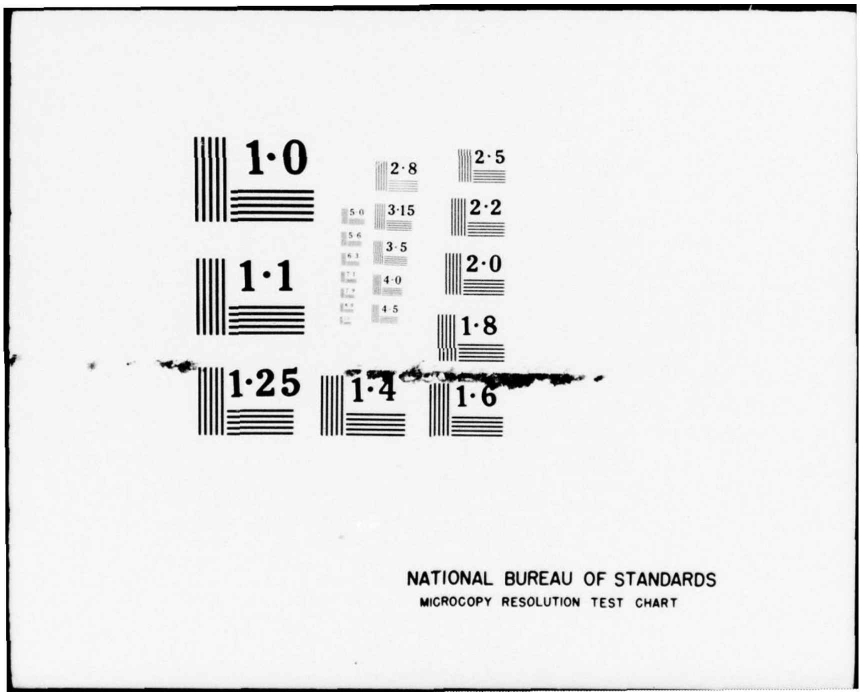
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ON THE SPEED OF PROPAGATION OF WAVES IN

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by

K.N. SAWYERS and R.S. RIVLIN

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On the Speed of Propagation of Waves

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ABSTRACT

The secular equation is obtained for small amplitude waves propagated in an arbitrary direction in a body of incompressible isotropic elastic material subjected to a pure homogeneous deformation. Conditions are obtained that the wave speeds be real.

1. Introduction

In a previous paper [1], we have considered the propagation of plane sinusoidal waves of infinitesimal amplitude in an incompressible isotropic elastic material which is subjected to an initial static pure homogeneous deformation. There the secular equation was obtained for waves propagated in an arbitrary direction in a principal plane of the pure homogeneous deformation. The requirement that the wave speeds be real for all such directions of propagation led to certain necessary and sufficient conditions on the form of the strain-energy function W. These conditions are expressed by equations (4.6) below with the notation of (3.13). The conditions $(4.6)_1$ were previously given by Baker and Ericksen [2], while the conditions $(4.6)_2$ were not, we believe, previously known.

In another paper [3], we conjectured that more stringent conditions on W than those in (4.6) might arise from the requirement that the wave speeds be real for arbitrary direction of propagation, not necessarily in a principal plane of the static pure homogeneous deformation. It is argued here (§5) that this is certainly not true in the case when two of the principal extension ratios are equal.

In the present paper we obtain the secular equation for waves propagated in an arbitrary direction in the deformed material. From it we easily obtain necessary and sufficient conditions that the wave speeds be real. However, these conditions involve not only the principal extension ratios and the derivatives of W with respect to the strain invariants, but also the direction of wave propagation. We have not, so far, been able to obtain the necessary and sufficient conditions in a form which is independent of this direction for arbitrary strain-energy functions and arbitrary values of the principal extension ratios associated with the pure homogeneous deformation.

However, in §6, it is shown that for strain-energy functions of the Mooney-Rivlin type, the Baker-Ericksen conditions provide necessary and sufficient conditions for the velocities of waves propagated in an arbitrary direction to be real. Essentially the same result was previously obtained by Ericksen [4] in his study of the propagation of a second-order discontinuity in an incompressible isotropic elastic material.

Finally, in §7 it is shown that if w depends on only one of the two strain invariants i_1 and i_2, defined in (2.4), then the restrictions expressed by equations (4.6) are necessary and sufficient that the wave speeds be real for arbitrary direction of propagation.

2. Basic equations

We consider a body of incompressible isotropic elastic material subjected to a pure homogeneous deformation with extension ratios $\lambda_1, \lambda_2, \lambda_3$ and principal directions parallel to the axes of a rectangular cartesian coordinate system x. In this deformation, a particle which initially has vector position $\underset{\sim}{\xi}$ with respect to the origin of the system x moves to vector position $\underset{\sim}{X}$, where

$$\underset{\sim}{X} = \underset{\sim}{\Lambda}\underset{\sim}{\xi} \ , \quad \underset{\sim}{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \tag{2.1}$$

and

$$\det\underset{\sim}{\Lambda} = \lambda_1\lambda_2\lambda_3 = 1 \ . \tag{2.2}$$

We now assume that a plane sinusoidal wave of small amplitude propagates in the deformed material. With the usual complex notation, the vector displacement $\underset{\sim}{u}$ may be written in the form

$$\underset{\sim}{u} = \underset{\sim}{U} \exp\iota[\omega(S\underset{\sim}{\ell}\cdot\underset{\sim}{X}-t)] \ , \tag{2.3}$$

where $\underset{\sim}{U}$ is a constant vector, ω is the angular frequency of the wave, S is its complex slowness and $\underset{\sim}{\ell}$ is a unit vector in the direction of propagation.

For an incompressible isotropic elastic material, subjected to a deformation in which a particle initially at $\underset{\sim}{\xi}$ moves to $\underset{\sim}{x}$ at time t, the strain-energy w, measured per unit volume, is expressible as a function of i_1 and i_2, thus

$$w = w(i_1, i_2) \ , \tag{2.4}$$

where i_1 and i_2 are defined by

$$i_1 = \operatorname{tr} \underset{\sim}{c} \ , \quad i_2 = \tfrac{1}{2}[(\operatorname{tr} \underset{\sim}{c})^2 - \operatorname{tr} \underset{\sim}{c}^2] \ , \tag{2.5}$$

and $\underset{\sim}{c} = \| c_{ij} \|$ is the Fingerstrain matrix, referred to the system x , defined by

$$c_{ij} = x_{i,\alpha} x_{j,\alpha} \ . \tag{2.6}$$

x_i and ξ_α are the components of $\underset{\sim}{x}$ and $\underset{\sim}{\xi}$ in the system x and $,\alpha$ denotes differentiation with respect to ξ_α . Correspondingly, the Cauchy stress matrix $\underset{\sim}{\sigma} = \| \sigma_{ij} \|$, referred to the system x , is given by

$$\underset{\sim}{\sigma} = 2[(w_1 + i_1 w_2)\underset{\sim}{c} - w_2 \underset{\sim}{c}^2] - p\underset{\sim}{\delta} \ , \tag{2.7}$$

where p is an arbitrary hydrostatic pressure, $\underset{\sim}{\delta}$ is the unit matrix, and the notation

$$w_1 = \partial w/\partial i_1 \ , \quad w_2 = \partial w/\partial i_2 \tag{2.8}$$

is used. Since the material considered is incompressible, the deformation gradients $x_{i,\alpha}$ must satisfy the condition

$$\det |x_{i,\alpha}| = 1 \ . \tag{2.9}$$

We now assume that

$$\underset{\sim}{x} = \underset{\sim}{X} + Re\,\underset{\sim}{u} = \underset{\sim}{\Lambda}\underset{\sim}{\xi} + Re\,\underset{\sim}{u}\,, \tag{2.10}$$

where $\underset{\sim}{u}$ is given by (2.3) and is assumed to be sufficiently small so that we can linearize in it. We then obtain [1] from (2.7) and the equations of motion, the following equation

$$S^2(Q_{ij} - Q\delta_{ij})\ell_j = \rho U_i\,, \tag{2.11}$$

where Q is an arbitrary constant and Q_{ij} is defined by

$$Q_{ij} = R_{ijk}U_k\,, \tag{2.12}$$

with*

$$\begin{aligned} R_{ABC} = 2\{&[W_1 + (I_1 - \lambda_A^2 - \lambda_B^2)W_2](\lambda_A^2\ell_A\delta_{BC} + \lambda_B^2\ell_B\delta_{AC}) \\ &+ 2\lambda_A^2\lambda_C^2\ell_C[W_2 + W_{11} + (2I_1 - \lambda_A^2 - \lambda_C^2)W_{12} \\ &+ (I_1 - \lambda_A^2)(I_1 - \lambda_C^2)W_{22}]\delta_{AB}\}\,. \end{aligned} \tag{2.13}$$

In (2.13) the notation

$$\begin{aligned} &I_1 = i_1\big|_{\underset{\sim}{u}=0}\,, \quad I_2 = i_2\big|_{\underset{\sim}{u}=0}\,, \\ &W_1 = \frac{\partial w}{\partial i_1}\bigg|_{\underset{\sim}{u}=0}\,, \quad W_2 = \frac{\partial w}{\partial i_2}\bigg|_{\underset{\sim}{u}=0}\,, \\ &W_{11} = \frac{\partial^2 w}{\partial i_1^2}\bigg|_{\underset{\sim}{u}=0}\,, \quad \text{etc.} \end{aligned} \tag{2.14}$$

is used.

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* The summation convention is not applied to upper case Latin subscripts.

In a similar manner, we obtain from (2.9), (2.10) and (2.3),

$$U_i \ell_i = 0 . \tag{2.15}$$

In the previous paper [1], we used (2.11) and (2.15) to obtain the secular equation for S in the particular case when the wave propagates in a principal plane of the pure homogeneous deformation, i.e. when either ℓ_1, ℓ_2, or ℓ_3 is zero. In the next section, we obtain the secular equation without this restriction.

3. The secular equation

We can eliminate Q from equation (2.11) by multiplying it throughout by $\varepsilon_{mni}\ell_n$ where ε_{mni} is the alternating symbol. We then obtain

$$\varepsilon_{mni}\sigma Q_{ij}\ell_n\ell_j = \varepsilon_{mni}\ell_n U_i \quad , \tag{3.1}$$

where we have introduced the notation

$$\sigma = S^2/\rho \ . \tag{3.2}$$

Introducing (2.12) into (3.1), we obtain

$$\bar{R}_{mk}U_k = 0 \tag{3.3}$$

where

$$\bar{R}_{mk} = \varepsilon_{mni}(\sigma R_{ijk}\ell_j - \delta_{ik})\ell_n \quad . \tag{3.4}$$

We note that the three equations represented by (3.3) are not linearly independent, since $\bar{R}_{mk}\ell_m$ is identically zero. Taking $m = 1$ and 2 in (3.3), we have two equations

$$\bar{R}_{1k}U_k = 0 \ , \quad \bar{R}_{2k}U_k = 0 \quad . \tag{3.5}$$

The necessary and sufficient condition for (3.5) and (2.15) to yield a non-trivial solution for U_k is

$$\varepsilon_{ijk}\bar{R}_{1j}\bar{R}_{2k}\ell_i = 0 \quad . \tag{3.6}$$

With (3.4) this can be rewritten as

$$\bar{\alpha}\sigma^2 - \bar{\beta}\sigma + \bar{\gamma} = 0 \quad , \tag{3.7}$$

where

$$\bar{\alpha} = \varepsilon_{ijk}\varepsilon_{1mn}\varepsilon_{2rs}R_{npj}R_{sqk}\ell_p\ell_q\ell_m\ell_r\ell_i \quad ,$$

$$\bar{\beta} = \varepsilon_{ijk}\varepsilon_{1mn}\varepsilon_{2rs}(R_{npj}\ell_p\delta_{sk} + R_{sqk}\ell_q\delta_{nj})\ell_m\ell_r\ell_i \quad , \tag{3.8}$$

$$\bar{\gamma} = \varepsilon_{ijk}\varepsilon_{1mj}\varepsilon_{2rk}\ell_i\ell_m\ell_r \quad .$$

After some algebraic manipulation, we obtain

$$\bar{\alpha} = \ell_3\varepsilon_{ijk}\ell_i\ell_p\ell_q(\ell_1R_{2pj}R_{3qk} + \ell_2R_{3pj}R_{1qk} + \ell_3R_{1pj}R_{2qk}) \quad ,$$

$$\bar{\beta} = \ell_3\ell_p(R_{kpk} - \ell_k\ell_rR_{kpr}) \quad , \tag{3.9}$$

$$\bar{\gamma} = \ell_3 \quad .$$

Introducing (3.9) into (3.7), we obtain, provided that* $\ell_3 \neq 0$,

$$\alpha\sigma^2 - \beta\sigma + 1 = 0 \quad , \tag{3.10}$$

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* If $\ell_3 = 0$, we take $m = 2$ and 3, or 3 and 1, in (3.4) to obtain another pair of equations instead of (3.5). Then, by a similar procedure to that used above, we arrive at (3.10) with (3.12).

where

$$\alpha = \sum_{\substack{A,B,C,\\P,Q}} \epsilon_{ABC}\ell_A\ell_P\ell_Q(\ell_1R_{2PB}R_{3QC} + \ell_2R_{3PB}R_{1QC} + \ell_3R_{1PB}R_{2QC}) \tag{3.11}$$

$$\beta = \sum_{A,B,C} \ell_B(R_{ABA} - \ell_A\ell_CR_{ABC})$$

Now, introducing (2.13) into (3.11), we obtain

$$\begin{aligned}\alpha = {} & 4(\lambda_1^2\ell_1^2+..)\{(\lambda_1^2\ell_1^2K_2K_3 +..) \\ & + W_1[\ell_2^2\ell_3^2(\lambda_2^2-\lambda_3^2)^2M_1 +..]\} \\ & + 4W_2[\lambda_2^2\lambda_3^2\ell_2^2\ell_3^2(\lambda_2^2-\lambda_3^2)^2M_1 +..] \\ & - 16\ell_1^2\ell_2^2\ell_3^2(\lambda_2^2-\lambda_3^2)^2(\lambda_3^2-\lambda_1^2)^2(\lambda_1^2-\lambda_2^2)^2(W_{12}^2-W_{11}W_{22}) , \\ \beta = {} & 2\{[K_1(\lambda_2^2\ell_2^2 + \lambda_3^2\ell_3^2) +..] \\ & + [(\lambda_2^2-\lambda_3^2)^2\ell_2^2\ell_3^2M_1 +..]\} ,\end{aligned} \tag{3.12}$$

where the dots denote terms obtained from those shown by cyclic permutation of the subscripts 1,2,3 on the λ's, ℓ's, K's and M's, and K_A and M_A are defined by

$$\begin{aligned}K_A &= W_1 + \lambda_A^2W_2 , \\ M_A &= 2(W_{11}+ 2\lambda_A^2W_{12}+ \lambda_A^4W_{22}) .\end{aligned} \tag{3.13}$$

From (3.12) it follows, after somewhat lengthy algebraic manipulation, that

$$\begin{aligned}\beta^2-4\alpha = {} & 4W_2^2\{[\lambda_1^4\ell_1^4(\lambda_2^2-\lambda_3^2)^2 + \ldots] \\ & + 2[\lambda_2^2\lambda_3^2\ell_2^2\ell_3^2(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2) + \ldots]\} \\ & + 8W_2\{[(\lambda_2^2-\lambda_3^2)^2\ell_2^2\ell_3^2M_1\{\lambda_2^2(\lambda_1^2-\lambda_3^2)(\ell_1^2+\ell_2^2) \\ & \qquad + \lambda_3^2(\lambda_1^2-\lambda_2^2)(\ell_1^2+\ell_3^2)\}] + \ldots\} \\ & + 4\{[(\lambda_2^2-\lambda_3^2)^2\ell_2^2\ell_3^2M_1 + \ldots]^2 \\ & + 16\ell_1^2\ell_2^2\ell_3^2(\lambda_2^2-\lambda_3^2)^2(\lambda_3^2-\lambda_1^2)^2(\lambda_1^2-\lambda_2^2)^2(W_{12}^2-W_{11}W_{22})\} \end{aligned} \tag{3.14}$$

It is of interest to note that $\beta^2 - 4\alpha$ is independent of W_1.

The necessary and sufficient conditions for the secular equation (3.10) to yield two positive roots for σ, i.e. for S^2, are

$$\alpha > 0 \ , \quad \beta > 0 \ , \quad \beta^2 \geq 4\alpha \ . \tag{3.15}$$

The remainder of this paper is concerned with the conditions which must be satisfied by w so that (3.15) shall be satisfied.

4. Propagation in a principal plane

If the direction of propagation of the wave lies in the principal plane normal to the 3-direction, so that $\underset{\sim}{\ell} = (\ell_1, \ell_2, 0)$, then equations (3.12) can be written as

$$\begin{aligned} \alpha &= 4(\lambda_1^2\ell_1^2K_2 + \lambda_2^2\ell_2^2K_1)\{K_3(\lambda_1\ell_1^2 - \lambda_2\ell_2^2)^2 \\ &\quad + \ell_1^2\ell_2^2(\lambda_1+\lambda_2)^2[K_3 + (\lambda_1-\lambda_2)^2M_3]\} , \\ \beta &= 2\{\lambda_1^2\ell_1^2K_2 + \lambda_2^2\ell_2^2K_1 + K_3(\lambda_1\ell_1^2 - \lambda_2\ell_2^2)^2 \\ &\quad + \ell_1^2\ell_2^2(\lambda_1+\lambda_2)^2[K_3 + (\lambda_1-\lambda_2)^2M_3]\} \end{aligned} \tag{4.1}$$

and equation (3.14) can be written as

$$\beta^2 - 4\alpha = 4\{W_2[\lambda_1^2\ell_1^2(\lambda_3^2-\lambda_2^2) + \lambda_2^2\ell_2^2(\lambda_3^2-\lambda_1^2) + \ell_1^2\ell_2^2(\lambda_1^2-\lambda_2^2)^2M_3\}^2 \tag{4.2}$$

We note that the weak inequality $(3.15)_3$ is automatically satisfied.

Again, if the direction of propagation of the wave is a principal direction for the pure homogeneous deformation, say the direction (1,0,0), then equations (4.1) become

$$\alpha = 4\lambda_1^4K_2K_3 , \quad \beta = 2\lambda_1^2(K_2+K_3) . \tag{4.3}$$

Analogous expressions for α and β may be written in the cases when the direction of propagation is (0,1,0) and (0,0,1) and it is then seen that necessary conditions for $\alpha > 0$ and $\beta > 0$ are

$$K_A > 0 \quad (A = 1,2,3) . \tag{4.4}$$

These conditions are known as the Baker-Ericksen conditions [2].

Returning to $(4.1)_1$, we see that provided (4.4) is satisfied, the necessary and sufficient condition for $\alpha > 0$ is

$$K_3 + (\lambda_1 - \lambda_2)^2 M_3 > 0 \ . \tag{4.5}$$

If (4.4) and (4.5) are satisfied then, from $(4.1)_2$, $\beta > 0$. Analogous arguments based on the expressions for α and β obtained by taking $\underset{\sim}{\ell} = (0, \ell_2, \ell_3)$ and $(\ell_1, 0, \ell_3)$ in (3.12), lead to the conclusion that the necessary and sufficient conditions for (3.15) to be satisfied for all $\underset{\sim}{\ell}$ parallel to a principal plane are

$$K_A > 0 \quad \text{and} \quad (\lambda_B - \lambda_C)^2 M_A + K_A > 0 \ , \tag{4.6}$$

where A, B, C is a cyclic permutation of 1,2,3. This result was previously obtained by a slightly different path in [1].

It will now be shown that if the conditions (4.6) are satisfied by w, then $\beta > 0$ for <u>all</u> $\underset{\sim}{\ell}$. To see this we rewrite $(3.12)_2$ in the form

$$\begin{aligned} \tfrac{1}{2}\beta = {} & [K_1\{(\lambda_2\ell_2^2 - \lambda_3\ell_3^2)^2 + \ell_1^2(\lambda_2^2\ell_2^2 + \lambda_3^2\ell_3^2)\}] \\ & + [\ \dots\ ] + [\ \dots\ ] + b \ , \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} b = {} & [(\lambda_2 + \lambda_3)^2 \ell_2^2 \ell_3^2 \{(\lambda_2 - \lambda_3)^2 M_1 + K_1\}] \\ & + [\ \dots\ ] + [\ \dots\ ] \ . \end{aligned} \tag{4.8}$$

5. Two principal extension ratios equal

In this section we will show that if two of the principal extension ratios are equal, then the conditions (4.6) imply that the conditions (3.15) are satisfied for arbitrary $\underset{\sim}{\ell}$.

Let

$$\lambda_2 = \lambda_3 = \lambda \ , \quad \text{say} \ . \tag{5.1}$$

Then, from (2.2)

$$\lambda_1 = \lambda^{-2} \ , \tag{5.2}$$

and the inequalities (4.6) become

$$\begin{gathered} K_1 > 0 \ , \quad K > 0 \ , \\ (\lambda^3 - 1)^2 M + \lambda^4 K > 0 \ , \end{gathered} \tag{5.3}$$

where, from (3.13),

$$\begin{gathered} K_1 = W_1 + \lambda^{-4} W_2 \ , \ K = K_2 = K_3 = W_1 + \lambda^2 W_2 \ , \\ M = M_2 = M_3 = 2(W_{11} + 2\lambda^2 W_{12} + \lambda^4 W_{22}) \ . \end{gathered} \tag{5.4}$$

With (5.4), we obtain from (3.12)

$$\alpha = 4\lambda^{-12}\{\ell_1^2K + \lambda^6(\ell_2^2+\ell_3^2)K_1\}\{\lambda^4[\lambda^3(\ell_2^2+\ell_3^2) - \ell_1^2]^2K$$
$$+ (\lambda^3+1)^2\ell_1^2(\ell_2^2+\ell_3^2)[(\lambda^3-1)^2M + \lambda^4K]\} ,$$
$$\beta = 2\{(\ell_2^2+\ell_3^2)[(K_1+K)\lambda^2 + M\lambda^{-8}(\lambda^6-1)^2\ell_1^2]$$
$$+ 2K\lambda^{-4}\ell_1^2\} . \tag{5.5}$$

It follows from (5.4) and (5.5) that

$$\beta^2-4\alpha = 4\lambda^{-16}(\lambda^6-1)^2(\ell_2^2+\ell_3^2)^2[\lambda^6W_2 + (\lambda^6-1)\ell_1^2M]^2 . \tag{5.6}$$

It is evident that if the conditions (5.3) are satisfied, then the expressions for α,β and $\beta^2-4\alpha$ given in (5.5) and (5.6) satisfy (3.15).

This result, that if two of the principal extension ratios are equal, the conditions (4.6) imply (3.15) for arbitrary $\underset{\sim}{\ell}$, is, of course, by no means unexpected. For the conditions (4.6) are the conditions that (3.15) be valid for all $\underset{\sim}{\ell}$ parallel to a principal plane. If two of the extension ratios are equal then _any_ direction is parallel to a principal plane.

It should be noted that even if we restrict ourselves to underlying pure homogeneous deformations for which two of the principal extension ratios are equal, we cannot ensure that the velocities of wave propagation shall be real for all propagation directions by imposing restrictions on w beyond those implied by (5.3). For, we can choose a direction of propagation for which $\ell_1^2 = 1$ and obtain from $(5.5)_2$, $\beta = 4K\lambda^{-4}$, so that the conditions $\beta > 0$ and $K > 0$ are identical. Again, by choosing $\ell_1 = 0$, $(5.5)_1$ becomes $\alpha = 4\lambda^4 K_1 K$, which, with $K > 0$ and the condition $\alpha > 0$, yields $K_1 > 0$. Finally, it is possible to choose a direction of propagation for which $\ell_1^2 = \lambda^3(\ell_2^2+\ell_3^2)$, and it follows from $(5.5)_1$, with $K_1 > 0$ and $K > 0$, that $\alpha > 0$ only if $(5.3)_3$ is satisfied.

In a previous paper [3], Sawyers and Rivlin conjectured that the conditions (4.2), which are the necessary and sufficient conditions for the wave velocities to be real for directions of propagation parallel to a principal plane, might be replaced by the stronger conditions

$$K_A > 0 , \quad (\lambda_A + \lambda_B)^2 M_C + K_C \geq 0 , \tag{5.7}$$

where A,B,C is a cyclic permutation of 1,2,3, if we wish to ensure that the wave velocities be real for arbitrary directions of propagation. That this is not true, in general, is evidenced by the above discussion, and also from the results given in §7 below.

6. The Mooney-Rivlin material

We now consider a material for which the strain-energy function is given by

$$w = C_1(i_1-3) + C_2(i_2-3) \tag{6.1}$$

where C_1 and C_2 are constants. Then,

$$W_1 = C_1\ , \quad W_2 = C_2\ , \quad W_{11} = W_{22} = W_{12} = 0\ . \tag{6.2}$$

It follows from (3.13) that

$$K_A = C_1 + \lambda_A^2 C_2\ , \quad M_A = 0\ . \tag{6.3}$$

Then, from (3.12),

$$\begin{aligned} \alpha &= 4(\lambda_1^2\ell_1^2 + \ ..\ )\{\lambda_1^2\ell_1^2 K_2K_3 + \ ..\ \}\ , \\ \beta &= 2[K_1(\lambda_2^2\ell_2^2 + \lambda_3^2\ell_3^2) + \ ..]\ . \end{aligned} \tag{6.4}$$

We shall assume that the underlying pure homogeneous deformation to which the body is subjected is such that the three principal extension ratios are all different*. We may assume, without loss of generality, that $\lambda_1 > \lambda_2 > \lambda_3$. It follows from (6.4) and $(6.3)_1$ that

* If this is not the case, the analysis of the preceding section is applicable.

$$\beta^2-4\alpha = 4C_2^2\{[\lambda_1^2\ell_1^2(\lambda_2^2-\lambda_3^2) + \lambda_2^2\ell_2^2(\lambda_1^2-\lambda_3^2) - \lambda_3^2\ell_3^2(\lambda_1^2-\lambda_2^2)]^2$$

$$+ 4\lambda_2^2\lambda_3^2\ell_2^2\ell_3^2(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)\} \ . \tag{6.5}$$

From (6.4) it follows that the conditions $\alpha > 0$, $\beta > 0$ are satisfied for all $\underset{\sim}{\ell}$ if and only if $K_A > 0$ $(A=1,2,3)$. It is, of course, evident from (6.5) that the condition $\beta^2-4\alpha \geq 0$ is automatically satisfied.

7. Two additional special cases

In this section we shall discuss the cases when w is a function of i_1 only or of i_2 only.

If w is independent of i_2, so that

$$w = f(i_1) , \tag{7.1}$$

then equations (3.13) become

$$K_A = f' , \quad M_A = 2f'' \quad (A=1,2,3) , \tag{7.2}$$

where the prime denotes differentiation with respect to i_1. The conditions (4.6) then become

$$f' > 0 , \quad 2(\lambda_A - \lambda_B)^2 f'' + f' > 0 . \tag{7.3}$$

where $AB = 23, 31, 12$.

With (7.1) and (7.2), we obtain from $(3.12)_1$ and (3.14)

$$\begin{aligned} \alpha &= 4(\lambda_1^2 \ell_1^2 + \,..\,) f' \{ (\lambda_1^2 \ell_1^2 + \,..\,) f' \\ &\qquad + 2f'' [(\lambda_2^2 - \lambda_3^2)^2 \ell_2^2 \ell_3^2 + \,..\,] \} , \\ \beta^2 - 4\alpha &= 16[(\lambda_2^2 - \lambda_3^2)^2 \ell_2^2 \ell_3^2 + \,..\,]^2 (f'')^2 . \end{aligned} \tag{7.4}$$

It is evident that $\beta^2 - 4\alpha \geq 0$ for all forms of f .

We shall now show that $\alpha > 0$ for all functions f which satisfy (7.3). It is, of course, evident that the conditions (7.3) are satisfied for all functions f for which $f' > 0$ and $f'' \geq 0$ and, for all such functions, $\alpha > 0$. We accordingly consider only functions f for which $f' > 0$ and $f'' < 0$.

Let us assume* that $\lambda_1 > \lambda_2 > \lambda_3$. Then, the condition

$$2(\lambda_1-\lambda_3)^2 f'' + f' > 0 \tag{7.5}$$

implies the remaining two conditions $(7.3)_2$. We may rewrite (7.5) as

$$2(\lambda_1-\lambda_3)^2 f'' + f' = \varepsilon \ , \quad \varepsilon > 0 \ . \tag{7.6}$$

Then, with $(7.6)_1$, equation $(7.4)_1$ becomes

$$\alpha = \frac{4(\lambda_1^2 \ell_1^2 + ..)}{(\lambda_1-\lambda_3)^2} \ f'\{Ff' + \varepsilon[(\lambda_2^2-\lambda_3^2)^2 \ell_2^2 \ell_3^2 + ..]\} \ , \tag{7.7}$$

where

$$F = (\lambda_1-\lambda_3)^2(\lambda_1^2\ell_1^2 + \ ..) - [(\lambda_2^2-\lambda_3^2)^2 \ell_2^2 \ell_3^2 + ..] \ . \tag{7.8}$$

We now show that $F \geq 0$. Since $\ell_2^2 = 1-\ell_1^2-\ell_3^2$, we can rewrite (7.8) as

$$\begin{aligned} F = \lambda_2^2(\lambda_1-\lambda_3)^2 &+ (\lambda_1^2-\lambda_2^2)[(\lambda_1-\lambda_3)^2 - (\lambda_1^2-\lambda_2^2)]\ell_1^2 \\ &- (\lambda_2^2-\lambda_3^2)[(\lambda_1-\lambda_3)^2 + (\lambda_2^2-\lambda_3^2)]\ell_3^2 \\ &+ [(\lambda_1^2-\lambda_2^2)\ell_1^2 - (\lambda_2^2-\lambda_3^2)\ell_3^2]^2 \ . \end{aligned} \tag{7.9}$$

Regarding F as a function of ℓ_1^2 and ℓ_3^2 , we see that if F has a minimum, it occurs when $\partial F/\partial(\ell_1^2) = \partial F/\partial(\ell_3^2) = 0$, i.e. when

---

* The case when two of the λ's are equal is covered in §5.

$$(\lambda_1^2-\lambda_2^2)\ell_1^2-(\lambda_2^2-\lambda_3^2)\ell_3^2=-\tfrac{1}{2}[(\lambda_1-\lambda_3)^2-(\lambda_1^2-\lambda_2^2)] \;,$$
$$(\lambda_1^2-\lambda_2^2)\ell_1^2-(\lambda_2^2-\lambda_3^2)\ell_3^2=-\tfrac{1}{2}[(\lambda_1-\lambda_3)^2-(\lambda_3^2-\lambda_2^2)] \;. \qquad (7.10)$$

Since $\lambda_1 > \lambda_3$, equations (7.10) evidently yield no solution for ℓ_1^2 and ℓ_3^2 and, accordingly, F cannot assume a minimum value.

We find, from (7.9) and the fact that $\underset{\sim}{\ell}$ is a unit vector, that

$$\begin{aligned} F\big|_{\ell_1=0} &= (\lambda_1-\lambda_3)^2(\lambda_2\ell_2^2-\lambda_3\ell_3^2)^2 \\ &\quad + (\lambda_2+\lambda_3)^2(\lambda_1-\lambda_2)(\lambda_1+\lambda_2-2\lambda_3)\ell_2^2\ell_3^2 > 0 \;, \\ F\big|_{\ell_3=0} &= (\lambda_1-\lambda_3)^2(\lambda_1\ell_1^2-\lambda_2\ell_2^2)^2 \\ &\quad + (\lambda_1+\lambda_2)^2(\lambda_2-\lambda_3)(2\lambda_1-\lambda_2-\lambda_3)\ell_1^2\ell_2^2 > 0 \;, \\ F\big|_{\ell_2=0} &= (\lambda_1-\lambda_3)^2(\lambda_1\ell_1^2-\lambda_3\ell_3^2)^2 \geq 0 \;. \end{aligned} \qquad (7.11)$$

We note that $F\big|_{\ell_2=0} = 0$ if and only if $\lambda_1\ell_1^2-\lambda_3\ell_3^2 = 0$. The domain in the ℓ_1^2, ℓ_3^2 plane for which $\underset{\sim}{\ell}$ is a unit vector is the triangular domain bounded by the lines $\ell_1^2 = 0$, $\ell_3^2 = 0$ and $\ell_1^2+\ell_3^2 = 1$. It follows from (7.11) that $F \geq 0$ on the boundary of this domain. If $F < 0$ at any point in the interior of the domain, it follows (see, for example, [5]) that it must assume a minimum value in the interior of the domain. However, we have seen that it does not assume such a minimum. Accordingly, $F \geq 0$ at all points of the closed domain, i.e. for all $\underset{\sim}{\ell}$. It follows from (7.7) and (7.6) that $\alpha > 0$ for all $\underset{\sim}{\ell}$ if w has the form (7.1) and the conditions (4.6) are satisfied.

We now turn to the case when w is independent of i_1 and write

$$w = g(i_2) \tag{7.12}$$

Then, the conditions (4.6) become

$$g' > 0 \ , \quad 2(\lambda_A-\lambda_B)^2 \lambda_C^2 g''+ g' > 0 \ , \tag{7.13}$$

where the prime now denotes differentiation with respect to i_2 . With (7.12) and (3.13), we obtain from (3.12)

$$\begin{aligned} \alpha &= 4g'(g'\phi_4+ 2g''\phi_3) \quad \beta = 4(g'\phi_2+ g''\phi_1) \ , \\ \beta^2-4\alpha &= 16\{\phi_1^2(g'')^2 + 2(\phi_1\phi_2-\phi_3)g'g'' + (\phi_2^2-\phi_4)(g')^2\} \ , \end{aligned} \tag{7.14}$$

where

$$\begin{aligned} \phi_1 &= \lambda_1^4(\lambda_2^2-\lambda_3^2)^2\ell_2^2\ell_3^2 + \ldots \ , \\ \phi_2 &= \frac{1}{2}\{\lambda_1^2\ell_1^2(\lambda_2^2+\lambda_3^2) + \ldots\} \ , \\ \phi_3 &= \lambda_1^2(\lambda_2^2-\lambda_3^2)^2\ell_2^2\ell_3^2 + \ldots \ , \\ \phi_4 &= \lambda_1^2\ell_1^2 + \ldots \ . \end{aligned} \tag{7.15}$$

We may rewrite $(7.14)_2$ in the form

$$\beta^2-4\alpha = 16\{[\phi_1 g'' + (\phi_2- \frac{\phi_3}{\phi_1})g']^2 + (\frac{g'}{\phi_1})^2(2\phi_1\phi_2\phi_3- \phi_1^2\phi_4- \phi_3^2)\} \ . \tag{7.16}$$

After some lengthy calculation, we obtain

$$2\phi_1\phi_2\phi_3- \phi_1^2\phi_4- \phi_3^2 = \ell_1^2\ell_2^2\ell_3^2(\lambda_2^2-\lambda_3^2)^2(\lambda_3^2-\lambda_1^2)^2(\lambda_1^2-\lambda_2^2)^2 \geq 0. \tag{7.17}$$

Thus, the condition $\beta^2 - 4\alpha \geq 0$ is satisfied for all g.

It is evident that the conditions (7.13) are satisfied for all g for which $g' > 0$ and $g'' \geq 0$ and, $\alpha > 0$ for all such g. We accordingly consider only functions g for which $g' > 0$ and $g'' < 0$. Let us assume $\lambda_1 > \lambda_2 > \lambda_3$. Then, since, with (2.2),

$$\lambda_2^2(\lambda_1-\lambda_3)^2 > \lambda_1^2(\lambda_2-\lambda_3)^2 , \ \lambda_2^2(\lambda_1-\lambda_3)^2 > \lambda_3^2(\lambda_1-\lambda_2)^2 , \tag{7.18}$$

it follows that the condition

$$2\lambda_2^2(\lambda_1-\lambda_3)^2 g'' + g' > 0 , \tag{7.19}$$

implies the remaining two conditions $(7.13)_2$.

The condition (7.19) may be rewritten as

$$2\lambda_2^2(\lambda_1-\lambda_3)^2 g'' + g' = \varepsilon , \quad \varepsilon > 0 . \tag{7.20}$$

From $(7.20)_1$ and $(7.14)_1$, we obtain

$$\alpha = \frac{4g'}{\lambda_2^2(\lambda_1-\lambda_3)^2} \quad (Gg' + \varepsilon\phi_3) , \tag{7.21}$$

where

$$G = \lambda_2^2(\lambda_1-\lambda_3)^2\phi_4 - \phi_3 . \tag{7.22}$$

Writing $\ell_2^2 = 1 - \ell_1^2 - \ell_3^2$ in the expression (7.15) for ϕ_3 and ϕ_4 and regarding G as a function of ℓ_1^2 and ℓ_3^2 it follows that if G has a minimum, it occurs when

$$
\begin{aligned}
2\lambda_3^2(\lambda_1^2-\lambda_2^2)\ell_1^2- (\lambda_1^2+\lambda_3^2)(\lambda_2^2-\lambda_3^2)\ell_3^2 & \\
&= \lambda_3^2(\lambda_1^2-\lambda_2^2) - \lambda_2^2(\lambda_1-\lambda_3)^2 , \\
2\lambda_1^2(\lambda_3^2-\lambda_2^2)\ell_3^2- (\lambda_1^2+\lambda_3^2)(\lambda_2^2-\lambda_1^2)\ell_1^2 & \\
&= \lambda_1^2(\lambda_3^2-\lambda_2^2) - \lambda_2^2(\lambda_1-\lambda_3)^2 ,
\end{aligned} \tag{7.23}
$$

i.e. when

$$
\begin{aligned}
(\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_2^2)\ell_1^2 &= - \{\lambda_3^2(\lambda_1^2-\lambda_2^2) + 2\lambda_2\} , \\
(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)\ell_3^2 &= \lambda_1^2(\lambda_2^2-\lambda_3^2) - 2\lambda_2 .
\end{aligned} \tag{7.24}
$$

Since $\lambda_1 > \lambda_2 > \lambda_2$, the value of ℓ_1^2 given by (7.24) is negative and accordingly, G does not possess a minimum value for real values of $\underset{\sim}{\ell}$.

We find, from (7.22) and (7.15) and the fact that $\underset{\sim}{\ell}$ is a unit vector, that

$$
\begin{aligned}
G\big|_{\ell_1=0} &= (\lambda_1-\lambda_3)^2 \lambda_2^2(\lambda_2\ell_2^2-\lambda_3\ell_3^2)^2 \\
&\quad + \lambda_3(\lambda_1-\lambda_2)(\lambda_2+\lambda_3)^2(2\lambda_1\lambda_2-\lambda_1\lambda_3-\lambda_2\lambda_3)\ell_2^2\ell_3^2 > 0 , \\
G\big|_{\ell_3=0} &= (\lambda_1-\lambda_3)^2 \lambda_2^2(\lambda_1\ell_1^2-\lambda_2\ell_2^2)^2 \\
&\quad + \lambda_1(\lambda_2-\lambda_3)(\lambda_1+\lambda_2)^2(\lambda_1\lambda_2+\lambda_1\lambda_3-2\lambda_2\lambda_3)\ell_1^2\ell_2^2 > 0 , \\
G\big|_{\ell_2=0} &= \lambda_2^2(\lambda_1-\lambda_3)^2(\lambda_1\ell_1^2-\lambda_3\ell_3^2)^2 \geq 0 .
\end{aligned} \tag{7.25}
$$

We note that $G\big|_{\ell_2=0} = 0$ if and only if $\lambda_1\ell_1^2 = \lambda_3\ell_3^2$. By an argument similar to that used in showing that $F \geq 0$, it follows that $G \geq 0$ for all real $\underset{\sim}{\ell}$ and the value $G = 0$

is taken only when $\underset{\sim}{\ell}$ is perpendicular to the 2-direction. It then follows from (7.21) and (7.15), with (7.20), that $\alpha > 0$ for all $\underset{\sim}{\ell}$ if w has the form (7.12) and the conditions (4.6) are satisfied.

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The secular equation is obtained for small amplitude waves propagated in an arbitrary direction in a body of incompressible isotropic elastic material subjected to a pure homogeneous deformation. Conditions are obtained that the wave speeds be real.

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